

PROPERTIES OF OPERATORS RELATED TO A Γ_3 -CONTRACTION AND UNITARY INVARIANTS

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ABSTRACT. The closed symmetrized polydisc of dimension three is the set

$$\Gamma_3 = \{(z_1 + z_2 + z_3, z_1 z_2 + z_2 z_3 + z_3 z_1, z_1 z_2 z_3) : |z_i| \leq 1, i = 1, 2, 3\} \subseteq \mathbb{C}^3.$$

A triple of commuting operators for which Γ_3 is a spectral set is called a Γ_3 -contraction. For a Γ_3 -contraction (S_1, S_2, P) there are two unique operators A_1, A_2 such that

$$S_1 - S_2^* P = D_P A_1 D_P, \quad S_2 - S_1^* P = D_P A_2 D_P.$$

The operator pair (A_1, A_2) plays central role in determining the structure of a Γ_3 -contraction. We shall discuss various properties of the fundamental operator pairs of Γ_3 -contractions. For two operator pairs (A_1, A_2) and (B_1, B_2) we provide conditions under which there exists a Γ_3 -contraction (S_1, S_2, P) such that (A_1, A_2) is the fundamental operator pair of (S_1, S_2, P) and (B_1, B_2) is the fundamental operator pair of its adjoint (S_1^*, S_2^*, P^*) . We shall show that such fundamental operator pair plays pivotal role in determining a set of unitary invariants for the Γ_3 -contractions.

1. INTRODUCTION

A triple of commuting operator (S_1, S_2, P) defined on a Hilbert space \mathcal{H} for which the closed symmetrized tridisc

$$\Gamma_3 = \{(z_1 + z_2 + z_3, z_1 z_2 + z_2 z_3 + z_3 z_1, z_1 z_2 z_3) : |z_i| \leq 1, i = 1, 2, 3\} \subseteq \mathbb{C}^3.$$

is a spectral set is called a Γ_3 -contraction. In Theorem 4.8 of [9], the author of this article have shown that corresponding to every Γ_3 -contraction (S_1, S_2, P) there exist two unique operators A_1, A_2 on the space \mathcal{D}_P such that

$$S_1 - S_2^* P = D_P A_1 D_P \text{ and } S_2 - S_1^* P = D_P A_2 D_P.$$

Here P is a contraction and $D_P = (I - P^* P)^{\frac{1}{2}}$ and $\mathcal{D}_P = \overline{\text{Ran}} D_P$. This unique pair (A_1, A_2) plays central role in determining the failure of rational dilation on Γ_3 and also in discovering several cases when it succeeds. For such major contribution in determining the structure of a Γ_3 -contraction

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this unique pair was named the *fundamental operator pair*. In this note, we shall discuss various important properties of the fundamental operator pairs of Γ_3 -contractions. Indeed, in Section 3 we determine a necessary and sufficient condition for a pair (A_1, A_2) to become the fundamental operator pair of a given Γ_3 -contraction. Also in the same section we determine the conditions for admissibility of two pairs of operators to become the fundamental operator pairs of a Γ_3 -contraction and its adjoint. Given two operator pairs (A_1, A_2) and (B_1, B_2) on certain Hilbert spaces, we find conditions under which there exists a Γ_3 -contraction (S_1, S_2, P) such that (A_1, A_2) is the fundamental operator pair of (S_1, S_2, P) and (B_1, B_2) is the fundamental operator pair of its adjoint (S_1^*, S_2^*, P^*) . In Section 4, we determine a set of unitary invariants for pure Γ_3 -contractions. We mention here that a Γ_3 -contraction (S_1, S_2, P) is said to be *pure* if P is a pure contraction, that is, $P^{*n} \rightarrow 0$ as $n \rightarrow \infty$. These invariants include the associated fundamental operator pairs. In Section 2, we accumulate few results from the existing literature which we shall use in the sequel.

2. BRIEF LITERATURE

We recall from [11] a set of characterizations of the closed and open symmetrized tridisc.

Theorem 2.1. *For $x = (s_1, s_2, p) \in \mathbb{C}^3$ the following are equivalent*

- (1) $x \in \Gamma_3$ (respectively $\in \mathbb{G}_3$) ;
- (2) $3 - s_1 z - s_2 w + 3pzw \neq 0, \forall z, w \in \mathbb{D}$ (respectively, $\in \overline{\mathbb{D}}$);
- (3) $|3s_1 - 3\bar{s}_2 p| + |s_1 s_2 - 9p| \leq 9 - |s_2|^2$ (respectively, $< 9 - |s_2|^2$) ;
- (3') $|3s_2 - 3\bar{s}_1 p| + |s_1 s_2 - 9p| \leq 9 - |s_1|^2$ (respectively, $< 9 - |s_1|^2$) ;
- (4) $|s_1|^2 - |s_2|^2 + 9|p|^2 + 6|s_2 - \bar{s}_1 p| \leq 9$ (respectively, < 9) and $|s_2| \leq 3$ (respectively, < 3) ;
- (4') $-|s_1|^2 + |s_2|^2 + 9|p|^2 + 6|s_1 - \bar{s}_2 p| \leq 9$ (respectively, < 9) and $|s_1| \leq 3$ (respectively, < 3) ;
- (5) $|s_1|^2 + |s_2|^2 - 9|p|^2 + 2|s_1 s_2 - 9p| \leq 9$ (respectively, < 9) and $|p| \leq 1$ (respectively, < 1) ;
- (6) $|s_1 - \bar{s}_2 p| + |s_2 - \bar{s}_1 p| \leq 3(1 - |p|^2)$ (respectively, $< 3(1 - |p|^2)$) ;
- (7) there exist a 2×2 matrix $B = [b_{ij}]$ such that $\|B\| \leq 1$ (respectively, < 1) and $x = (3b_{11}, 3b_{22}, \det B)$;
- (8) $|p| \leq 1$ (respectively, < 1) and there exist $c_1, c_2 \in \mathbb{C}$ such that $|c_1| + |c_2| \leq 3$ (respectively, < 3) and

$$s_1 = c_1 + \bar{c}_2 p, \quad s_2 = c_2 + \bar{c}_1 p.$$

Definition 2.2. A pair of operators (F_1, F_2) is said to be *almost normal* if A_1, A_2 commute and $A_1^* A_1 - A_1 A_1^* = A_2^* A_2 - A_2 A_2^*$.

Definition 2.3. A Γ_3 -contraction (S_1, S_2, P) is said to be a Γ_3 -unitary if the Taylor joint spectrum $\sigma_T(S_1, S_2, P)$ lies within the distinguished boundary

$$b\Gamma_3 = \{(z_1 + z_2 + z_3, z_1 z_2 + z_2 z_3 + z_3 z_1, z_1 z_2 z_3) : |z_i| = 1, i = 1, 2, 3\}.$$

A Γ_3 -isometry is the restriction of a Γ_3 -unitary to a common invariant subspace.

The following result from [9] gives a model for pure Γ_3 -isometries.

Theorem 2.4. Let $(\hat{S}_1, \hat{S}_2, \hat{P})$ be a commuting triple of operators on a Hilbert space \mathcal{H} . If $(\hat{S}_1, \hat{S}_2, \hat{P})$ is a pure Γ_3 -isometry then there is a unitary operator $U : \mathcal{H} \rightarrow H^2(\mathcal{D}_{\hat{P}^*})$ such that

$$\hat{S}_1 = U^* T_\varphi U, \quad \hat{S}_2 = U^* T_\psi U \text{ and } \hat{P} = U^* T_z U,$$

where $\varphi(z) = F_1^* + F_2 z$, $\psi(z) = F_2^* + F_1 z$, $z \in \mathbb{D}$ and (F_1, F_2) is the fundamental operator pair of $(\hat{S}_1^*, \hat{S}_2^*, \hat{P}^*)$ such that

- (1) (F_1, F_2) is almost normal
- (2) $\|F_1 + F_2 z\|_{\infty, \mathbb{D}} \leq 3$.

Conversely, if F_1 and F_2 are two bounded operators on a Hilbert space E satisfying the above two conditions, then $(T_{F_1^* + F_2 z}, T_{F_2^* + F_1 z}, T_z)$ on $H^2(E)$ is a pure Γ_3 -isometry.

3. FUNDAMENTAL OPERATOR PAIRS: PROPERTIES AND ADMISSIBILITY

The fundamental operator pair (A_1, A_2) of a Γ_3 -contraction (S_1, S_2, P) is the unique operator pair defined on \mathcal{D}_P that satisfies

$$(3.1) \quad S_1 - S_2^* P = D_P A_1 D_P \text{ and } S_2 - S_1^* P = D_P A_2 D_P.$$

Here \mathcal{D}_P is the closure of the range of the operator $(I - P^* P)^{\frac{1}{2}}$. In [9], the author has proved that for any Γ_3 -contraction (S_1, S_2, P) there exists two unique operators A_1, A_2 such that (3.1) holds ([9], Theorem 4.8). We shall see in the next section that the fundamental operator pair plays central role in determining a set of unitary invariants for Γ_3 -contractions.

3.1. Properties of fundamental operator pair. In this subsection, we shall obtain few important properties of the fundamental operator pair corresponding to a Γ_3 -contraction. It is evident from Theorem 4.8 in [9] that a pair of operators (A_1, A_2) defined on \mathcal{D}_P is fundamental operator pair if and only if it satisfies the following two operator equations in X_1, X_2 :

$$S_1 - S_2^* P = D_P X_1 D_P ; \quad S_2 - S_1^* P = D_P X_2 D_P.$$

The next theorem is going to prove another characterization for the fundamental operator pair of a Γ_3 -contraction.

Theorem 3.1. A pair of operators (A_1, A_2) defined on \mathcal{D}_P is the fundamental operator pair of a Γ_3 -contraction (S_1, S_2, P) if and only if (A_1, A_2) satisfy the following pair of operator equations in X_1, X_2 :

$$D_P S_1 = X_1 D_P + X_2^* D_P P \text{ and } D_P S_2 = X_2 D_P + X_1^* D_P P.$$

Proof. First let (A_1, A_2) be the fundamental operator pair of (S_1, S_2, P) . Then

$$S_1 - S_2^*P = D_P F_1 D_P \text{ and } S_2 - S_1^*P = D_P F_2 D_P.$$

Now

$$\begin{aligned} D_P(F_1 D_P + F_2^* D_P P) &= (S_1 - S_2^*P) + (S_2^* - P^* S_1)P \\ &= (I - P^*P)S_1 \\ &= D_P^2 S_1. \end{aligned}$$

Therefore, if $J = D_P S_1 - F_1 D_P - F_2^* D_P P$ then $J : \mathcal{H} \rightarrow \mathcal{D}_P$ and $D_P J = 0$. Now

$$\langle Jh, D_P h' \rangle = \langle D_P Jh, h' \rangle = 0 \quad \text{for all } h, h' \in \mathcal{H}.$$

This shows that $J = 0$ and hence $F_1 D_P + F_2^* D_P P = D_P S_1$. The proof of $D_P S_2 = F_2 D_P + F_1^* D_P P$ is similar.

Conversely, let (X_1, X_2) be a pair of operators on \mathcal{D}_P such that

$$D_P S_1 = X_1 D_P + X_2^* D_P P \text{ and } D_P S_2 = X_2 D_P + X_1^* D_P P.$$

We need to show that $(X_1, X_2) = (F_1, F_2)$. Since we just proved that (F_1, F_2) satisfies the equation, we have

$$F_1 D_P + F_2^* D_P P = D_P S_1 = X_1 D_P + X_2^* D_P P.,$$

and

$$F_2 D_P + F_1^* D_P P = D_P S_2 = X_2 D_P + X_1^* D_P P.$$

and consequently

$$(X_1 - F_1)D_P + (X_2 - F_2)^* D_P P = (X_2 - F_2)D_P + (X_1 - F_1)^* D_P P = 0.$$

Let

$$Y_1 = X_1 - F_1, Y_2 = X_2 - F_2.$$

Then

$$(3.2) \quad Y_1 D_P + Y_2^* D_P P = Y_2 D_P + Y_1^* D_P P = 0.$$

To complete the proof, we need to show that $Y_1 = Y_2 = 0$. We have

$$\begin{aligned} &Y_1 D_P + Y_2^* D_P P = 0 \\ \text{or} \quad &Y_1 D_P = -Y_2^* D_P P \\ \text{or} \quad &D_P Y_1 D_P = -D_P Y_2^* D_P P \\ \text{or} \quad &D_P Y_1^* D_P = P^* D_P Y_1^* D_P P = P^{*2} D_P Y_1^* D_P P^2 = \dots \end{aligned}$$

We obtained the equalities in the last line by applying (3.2). Thus we have

$$(3.3) \quad D_P Y_1^* D_P = P^{*n} D_P Y_1^* D_P P^n$$

for all $n = 1, 2, \dots$. Now consider the series

$$\begin{aligned}
 \sum_{n=0}^{\infty} \|D_P P^n h\|^2 &= \sum_{n=0}^{\infty} \langle D_P P^n h, D_P P^n h \rangle \\
 &= \sum_{n=0}^{\infty} \langle P^{*n} D_P^2 P^n h, h \rangle \\
 &= \sum_{n=0}^{\infty} \langle P^{*n} (I - P^* P) P^n h, h \rangle \\
 &= \sum_{n=0}^{\infty} \langle (P^{*n} P^n - P^{*n+1} P^{n+1} h, h) \rangle \\
 &= \sum_{n=0}^{\infty} (\|P^n h\|^2 - \|P^{n+1} h\|^2) \\
 &= \|h\|^2 - \lim_{n \rightarrow \infty} \|P^n h\|^2.
 \end{aligned}$$

$$\|h\| \geq \|Ph\| \geq \|P^2 h\| \geq \dots \geq \|P^n h\| \geq \dots \geq 0.$$

So $\lim_{n \rightarrow \infty} \|P^n h\|^2$ exists. So the series is convergent. So $\lim_{n \rightarrow \infty} \|D_P P^n h\|^2 = 0$. So

$$\begin{aligned}
 \|D_P Y_1^* D_P h\| &= \|P^{*n} D_P Y_1^* D_P P^n h\| \text{ by (3.3)} \\
 &\leq \|P^{*n}\| \|D_P Y_1^*\| \|D_P P^n h\| \\
 &\leq \|D_P Y_1^*\| \|D_P P^n h\| \rightarrow 0.
 \end{aligned}$$

So $D_P Y_1^* D_P = 0$ and hence $Y_1 = 0$. Similarly one can show that $Y_2 = 0$. Hence the proof is complete. ■

The next few results will provide some interesting algebra of interplay between a Γ_3 -contraction and its fundamental operator pair.

Lemma 3.2. *Let (S_1, S_2, P) be a Γ_3 -contraction with commuting fundamental operator pair (A_1, A_2) . Then*

$$S_1^* S_1 - S_2^* S_2 = D_P (A_1^* A_1 - A_2^* A_2) D_P.$$

Proof. We have that (A_1, A_2) is a commuting pair satisfying

$$(3.4) \quad S_1 - S_2^* P = D_P A_1 D_P \text{ and } S_2 - S_1^* P = D_P A_2 D_P.$$

By the commutativity of S_1, S_2 we have $S_1^* S_2^* P = S_2^* S_1^* P$. Using (3.4) we have

$$S_1^* (S_1 - D_P A_1 D_P) = S_2^* (S_2 - D_P A_2 D_P),$$

which further implies that

$$\begin{aligned}
 S_1^* S_1 - S_2^* S_2 &= S_1^* D_P A_1 D_P - S_2^* D_P A_2 D_P \\
 &= (D_P A_1^* + P^* D_P A_2) A_1 D_P - (D_P A_2^* + P^* D_P A_1) A_2 D_P \\
 &= D_P (A_1^* A_1 - A_2^* A_2) D_P.
 \end{aligned}$$

To obtain the last two equalities we have used the commutativity of A_1, A_2 and the two identities

$$D_P S_1 = A_1 D_P + A_2^* D_P P \text{ and } D_P S_2 = A_2 D_P + A_1^* D_P P.$$

Hence we are done. ■

Lemma 3.3. *Let (S_1, S_2, P) be a Γ_3 -contraction on a Hilbert space \mathcal{H} and (A_1, A_2) and (B_1, B_2) be respectively the fundamental operator pairs of (S_1, S_2, P) and (S_1^*, S_2^*, P^*) respectively. Then*

$$D_P A_1 = (S_1 D_P - D_{P^*} B_2 P)|_{\mathcal{D}_P} \text{ and } D_P A_2 = (S_2 D_P - D_{P^*} B_1 P)|_{\mathcal{D}_P}.$$

Proof. Proving one of the identities is enough because the proof of the other is similar. For $h \in \mathcal{H}$, we have

$$\begin{aligned} (S_1 D_P - D_{P^*} B_2 P) D_P h &= S_1 (I - P^* P) h - (D_{P^*} B_2 D_{P^*}) P h \\ &= S_1 h - S_1 P^* P h - (S_2^* - S_1 P^*) P h \\ &= S_1 h - S_1 P^* P h - S_2^* P h + S_1 P^* P h \\ &= (S_1 - S_2^* P) h = (D_P A_1) D_P h. \end{aligned}$$

Hence the proof. ■

Lemma 3.4. *Let (S_1, S_2, P) be a Γ_3 -contraction on a Hilbert space \mathcal{H} and (A_1, A_2) and (B_1, B_2) be respectively the fundamental operator pairs of (S_1, S_2, P) and (S_1^*, S_2^*, P^*) respectively. Then*

$$P A_i = B_i^* P|_{\mathcal{D}_P}, \text{ for } i = 1, 2.$$

Proof. We shall prove only for $i = 1$, the proof for $i = 2$ is similar. Note that the operators on both sides are from \mathcal{D}_P to \mathcal{D}_{P^*} . Let $h, h' \in \mathcal{H}$ be any element. Then

$$\begin{aligned} &\langle (P A_1 - B_1^* P) D_P h, D_{P^*} h' \rangle \\ &= \langle D_{P^*} P A_1 D_P h, h' \rangle - \langle D_{P^*} B_1^* P D_P h, h' \rangle \\ &= \langle P (D_P A_1 D_P) h, h' \rangle - \langle (D_{P^*} B_1^* D_{P^*}) P h, h' \rangle \\ &= \langle P (S_1 - S_2^* P) h, h' \rangle - \langle (S_1 - P S_2^*) P h, h' \rangle \\ &= \langle (P S_1 - P S_2^* P - S_1 P + P S_2^* P) h, h' \rangle = 0. \end{aligned}$$

Hence the proof. ■

Lemma 3.5. *Let (S_1, S_2, P) be a Γ_3 -contraction on a Hilbert space \mathcal{H} and (A_1, A_2) and (B_1, B_2) be the fundamental operator pair of (S_1, S_2, P) and (S_1^*, S_2^*, P^*) respectively. Then*

$$\begin{aligned} (A_1^* D_P D_{P^*} - A_2 P^*)|_{\mathcal{D}_{P^*}} &= D_P D_{P^*} B_1 - P^* B_2^* \text{ and} \\ (A_2^* D_P D_{P^*} - A_1 P^*)|_{\mathcal{D}_{P^*}} &= D_P D_{P^*} B_2 - P^* B_1^*. \end{aligned}$$

Proof. For $h \in \mathcal{H}$, we have

$$\begin{aligned}
 & (A_1^* D_P D_{P^*} - A_2 P^*) D_{P^*} h \\
 &= A_1^* D_P (I - P P^*) h - A_2 P^* D_{P^*} h \\
 &= A_1^* D_P h - A_1^* D_P P P^* h - A_2 D_P P^* h \\
 &= A_1^* D_P h - (A_1^* D_P P + A_2 D_P) P^* h \\
 &= A_1^* D_P h - D_P S_2 P^* h \quad [\text{by Lemma (3.1)}] \\
 &= (S_1 D_P - D_{P^*} B_2 P)^* h - D_P S_2 P^* h \quad [\text{by Lemma 3.3}] \\
 &= D_P S_1^* h - P^* B_2^* D_{P^*} h - D_P S_2 P^* h \\
 &= D_P (S_1^* - S_2 P^*) h - P^* B_2^* D_{P^*} h \\
 &= D_P D_{P^*} B_1 D_{P^*} h - P^* B_2^* D_{P^*} h \\
 &= (D_P D_{P^*} B_1 - P^* B_2^*) D_{P^*} h.
 \end{aligned}$$

One can similarly prove the other relation. ■

Now we prove the main result of this subsection.

Theorem 3.6. *Let (A_1, A_2) be the fundamental operator pair of a Γ_3 -contraction (S_1, S_2, P) on a Hilbert space \mathcal{H} . And let (B_1, B_2) be the fundamental operator pair of the adjoint (S_1^*, S_2^*, P^*) . If $[A_1, A_2] = 0$ and P has dense range, then the following identities hold:*

- (i) $[A_1, A_1^*] = [A_2, A_2^*]$
- (ii) $[B_1, B_2] = 0$
- (iii) $[B_1, B_1^*] = [B_2, B_2^*]$.

Proof. (i) We have $D_P S_1 = A_1 D_P + A_2^* D_P P$ by Theorem 3.1. On multiplication by A_2 from left we obtain,

$$\begin{aligned}
 & A_2 D_P S_1 = A_2 A_1 D_P + A_2 A_2^* D_P P \\
 & \Rightarrow D_P A_2 D_P S_1 = D_P A_2 A_1 D_P + D_P A_2 A_2^* D_P P \\
 & \Rightarrow (S_2 - S_1^* P) S_1 = D_P A_2 A_1 D_P + D_P A_2 A_2^* D_P P \\
 & \Rightarrow S_2 S_1 - S_1^* S_1 P = D_P A_2 A_1 D_P + D_P A_2 A_2^* D_P P.
 \end{aligned}$$

Similarly, multiplying $D_P S_2 = A_2 D_P + A_1^* D_P P$ by A_1 from left we get

$$S_1 S_2 - S_2^* S_2 P = D_P A_1 A_2 D_P + D_P A_1 A_1^* D_P P.$$

Subtracting these two equations we get

$$\begin{aligned}
 & (S_1^* S_1 - S_2^* S_2) P = D_P [A_1, A_2] D_P + D_P (A_1 A_1^* - A_2 A_2^*) D_P P \\
 & \Rightarrow D_P (A_1^* A_1 - A_2^* A_2) D_P P = D_P [A_1, A_2] D_P + D_P (A_1 A_1^* - A_2 A_2^*) D_P P \\
 & \Rightarrow D_P ([A_1, A_1^*] - [A_2, A_2^*]) D_P P = 0 \quad [\text{since } [F_1, F_2] = 0.] \\
 & \Rightarrow D_P ([A_1, A_1^*] - [A_2, A_2^*]) D_P = 0 \quad [\text{since } \text{Ran } P \text{ is dense in } \mathcal{H}.] \\
 & \Rightarrow [A_1, A_1^*] = [A_2, A_2^*]
 \end{aligned}$$

The first implication follows by applying Lemma 3.2. This essentially completes the proof of part-(i) of the theorem.

- (ii) Again from Lemma 3.4, we have that $PA_i = B_i^*P|_{\mathcal{D}_P}$ for $i = 1, 2$.
So we have

$$\begin{aligned}
PA_1A_2D_P &= B_1^*PA_2D_P \\
\Rightarrow PA_2A_1D_P &= B_1^*PA_2D_P \\
\Rightarrow B_2^*B_1^*PD_P &= B_1^*B_2^*PD_P \\
\Rightarrow [B_1^*, B_2^*]D_{P^*}P &= 0 \\
\Rightarrow [B_1, B_2] &= 0
\end{aligned}$$

To obtain the implications we have used the commutativity of A_1, A_2 , the density of range of P in \mathcal{H} and Lemma 3.4. This completes the proof of part (ii) of the theorem.

- (iii) From Lemma 3.3, we have $D_PA_1 = (S_1D_P - D_{P^*}B_2P)|_{\mathcal{D}_P}$. Which gives after multiplying A_2D_P from right in both sides

$$\begin{aligned}
D_PA_1A_2D_P &= S_1D_PA_2D_P - D_{P^*}B_2PA_2D_P \\
\Rightarrow D_PA_1A_2D_P &= S_1(S_2 - S_1^*P) - D_{P^*}B_2B_2^*PD_P \text{ [applying Lemma 3.4.]} \\
\Rightarrow D_PA_1A_2D_P &= S_1S_2 - S_1S_1^*P - D_{P^*}B_2B_2^*PD_P.
\end{aligned}$$

Similarly, multiplying $D_PA_2 = (S_2D_P - D_{P^*}B_1P)|_{\mathcal{D}_P}$ by A_1D_P from right we get

$$D_PA_2A_1D_P = S_2S_1 - S_2S_2^*P - D_{P^*}B_1B_1^*PD_P.$$

Subtracting those two equations we get

$$\begin{aligned}
D_P[A_1, A_2]D_P &= D_{P^*}(B_1B_1^* - B_2B_2^*)D_{P^*}P - (S_1S_1^* - S_2S_2^*)P \\
\Rightarrow D_P[A_1, A_2]D_P &= D_{P^*}([B_1, B_1^*] - [B_2, B_2^*])D_{P^*}P \\
\Rightarrow D_{P^*}([B_1, B_1^*] - [B_2, B_2^*])D_{P^*}P &= 0 \\
\Rightarrow [B_1, B_1^*] &= [B_2, B_2^*]
\end{aligned}$$

We have used density of the range of P in \mathcal{H} and Lemma 3.2. Hence the proof is done. \blacksquare

Corollary 3.7. *Let (S_1, S_2, P) be a Γ_3 -contraction on a Hilbert space \mathcal{H} such that P is invertible. Let (A_1, A_2) and (B_1, B_2) be as in Theorem 3.6. Then $[A_1, A_2] = 0$ if and only if $[B_1, B_2] = 0$.*

Proof. Suppose that $[A_1, A_2] = 0$. Since P has dense range, by part (ii) of Theorem 3.6, we get $[B_1, B_2] = 0$. Conversely, let $[B_1, B_2] = 0$. Since P is invertible, P^* has dense range too. So applying Theorem 3.6 for the tetrablock contraction (S_1^*, S_2^*, P^*) , we get $[A_1, A_2] = 0$. \blacksquare

3.2. Admissibility of fundamental operator pair. Given two pairs of operators (A_1, A_2) and (B_1, B_2) defined on some certain Hilbert spaces. It is natural to ask when there exists a Γ_3 -contraction (S_1, S_2, P) such that (A_1, A_2) is the fundamental operator pair of (S_1, S_2, P) and (B_1, B_2) is the fundamental operator pair of its adjoint (S_1^*, S_2^*, P^*) . In this subsection, we shall answer this question. We begin with a lemma.

Lemma 3.8. *Let (A_1, A_2) be fundamental operator pair of a Γ_3 -contraction (S_1, S_2, P) and (B_1, B_2) be fundamental operators of the Γ_3 -contraction (S_1^*, S_2^*, P^*) . Then*

$$(3.5) \quad (A_1^* + A_2 z) \Theta_{P^*}(z) = \Theta_{P^*}(z) (B_1 + B_2^* z)$$

$$(3.6) \quad (A_2^* + A_1 z) \Theta_{P^*}(z) = \Theta_{P^*}(z) (B_2 + B_1^* z), \text{ for all } z \in \mathbb{D}.$$

Proof. We prove equation (3.5) only, proof of equation (3.6) is similar.

$$\begin{aligned} & (A_1^* + A_2 z) \Theta_{P^*}(z) \\ = & (A_1^* + A_2 z) \left(-P^* + \sum_{n=0}^{\infty} z^{n+1} D_P P^n D_{P^*} \right) \\ = & (-A_1^* P^* + \sum_{n=1}^{\infty} z^n A_1^* D_P P^{n-1} D_{P^*}) \\ & + (-z A_2 P^* + \sum_{n=2}^{\infty} z^n A_2 D_P P^{n-2} D_{P^*}) \\ = & -A_1^* P^* + z(A_1^* D_P D_{P^*} - A_2 P^*) \\ & + \sum_{n=2}^{\infty} z^n (A_1^* D_P P^{n-1} D_{P^*} + A_2 D_P P^{n-2} D_{P^*}) \\ = & -A_1^* P^* + z(A_1^* D_P D_{P^*} - A_2 P^*) + \sum_{n=2}^{\infty} z^n (A_1^* D_P P + A_2 D_P) P^{n-2} D_{P^*} \\ = & -P^* B_1 + z(D_P D_{P^*} B_1 - P^* B_2^*) + \sum_{n=2}^{\infty} z^n D_P S_2 P^{n-2} D_{P^*}. \end{aligned}$$

The last equality follows by using Theorem 3.1, Lemma 3.4 and Lemma 3.5. Also we have

$$\begin{aligned}
& \Theta_{P^*}(z)(B_1 + B_2^*z) \\
&= (-P^* + \sum_{n=0}^{\infty} z^{n+1} D_P P^n D_{P^*})(B_1 + B_2^*z) \\
&= (-P^* B_1 + \sum_{n=1}^{\infty} z^n D_P P^{n-1} D_{P^*} B_1) + (-z P^* B_2^* + \sum_{n=2}^{\infty} z^n D_P P^{n-2} D_{P^*} B_2^*) \\
&= -P^* B_1 + z(D_P D_{P^*} B_1 - P^* B_2^*) \\
&\quad + \sum_{n=2}^{\infty} z^n (D_P P^{n-1} D_{P^*} B_1 + D_P P^{n-2} D_{P^*} B_2^*) \\
&= -P^* B_1 + z(D_P D_{P^*} B_1 - P^* B_2^*) + \sum_{n=2}^{\infty} z^n D_P P^{n-2} (P D_{P^*} B_1 + D_P B_2^*) \\
&= -P^* B_1 + z(D_P D_{P^*} B_1 - P^* B_2^*) + \sum_{n=2}^{\infty} z^n D_P P^{n-2} S_2 D_{P^*} \\
&= -P^* B_1 + z(D_P D_{P^*} B_1 - P^* B_2^*) + \sum_{n=2}^{\infty} z^n D_P S_2 P^{n-2} D_{P^*}.
\end{aligned}$$

Hence $(A_1^* + A_2 z) \Theta_{P^*}(z) = \Theta_{P^*}(z)(B_1 + B_2^* z)$ for all $z \in \mathbb{D}$. Hence the proof. \blacksquare

Theorem 3.9. *Let (A_1, A_2) and (B_1, B_2) be respectively the fundamental operator pairs of a Γ_3 -contraction (S_1, S_2, P) and its adjoint (S_1^*, S_2^*, P^*) . Then*

$$(3.7) \quad (B_1^* + B_2 z) \Theta_P(z) = \Theta_P(z)(A_1 + A_2^* z) \text{ and}$$

$$(3.8) \quad (B_2^* + B_1 z) \Theta_P(z) = \Theta_P(z)(A_2 + A_1^* z) \text{ holds for all } z \in \mathbb{D}.$$

Conversely, let P be a pure contraction on a Hilbert space \mathcal{H} . Let $B_1, B_2 \in \mathcal{B}(\mathcal{D}_{P^})$ have numerical radii no greater than 3 and (B_1, B_2) is almost normal. Suppose $A_1, A_2 \in \mathcal{B}(\mathcal{D}_P)$ with numerical radii no greater than 3 are such that A_1, A_2, B_1, B_2 satisfy the equations (3.7) and (3.8). Then there exists a Γ_3 -contraction (S_1, S_2, P) such that (A_1, A_2) is the fundamental operator pair of (S_1, S_2, P) and (B_1, B_2) is the fundamental operator pair of (S_1^*, S_2^*, P^*) .*

Proof. We apply Lemma 3.8 to the Γ_3 -contraction (S_1^*, S_2^*, P^*) to obtain the forward implication.

For the converse, let us define $W : \mathcal{H} \rightarrow H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$ by

$$W(h) = \sum_{n=0}^{\infty} z^n \otimes D_{P^*} P^{*n} h \text{ for all } h \in \mathcal{H}.$$

It is evident that

$$\begin{aligned} \|Wh\|^2 &= \sum_{n=0}^{\infty} \|D_{P^*} P^{*n} h\|^2 = \sum_{n=0}^{\infty} \left(\|P^{*n} h\|^2 - \|P^{*(n+1)} h\|^2 \right) \\ &= \|h\|^2 - \lim_{n \rightarrow \infty} \|P^{*n} h\|^2. \end{aligned}$$

Therefore W is an isometry if P is a pure contraction. It is obvious that

$$W^*(z^n \otimes \xi) = P^n D_{P^*} \xi \text{ for all } \xi \in \mathcal{D}_{P^*} \text{ and } n \geq 0.$$

Also if M_z is the multiplication operator on the vectorial Hardy space $H^2(\mathcal{D}_{P^*})$, then we have

$$M_z^* W h = M_z^* \left(\sum_{n=0}^{\infty} z^n D_{P^*} P^{*n} h \right) = \sum_{n=0}^{\infty} z^n D_{P^*} P^{*(n+1)} h = W P^* h.$$

Therefore $M_z^* W = W P^*$. Almost normality of (B_1, B_2) implies that

$$(M_{B_1^*+B_2z}, M_{B_2^*+B_1z}, M_z)$$

is a commuting triple of bounded operators on $H_{\mathcal{D}_{P^*}}^2(\mathbb{D})$. It is evident from Theorem 2.4 that $(M_{B_1^*+B_2z}, M_{B_2^*+B_1z}, M_z)$ is a Γ_3 -isometry. Define $S_1 = W^* M_{B_1^*+B_2z} W$ and $S_2 = W^* M_{B_2^*+B_1z} W$. Equations (3.7) and (3.8) tells that $\text{Ran} M_{\Theta_P}$ is invariant under $M_{B_1^*+B_2z}$ and $M_{B_2^*+B_1z}$ which is same as saying that $\text{Ran} W = (\text{Ran} M_{\Theta_P})^\perp$ is invariant under $M_{B_1^*+B_2z}^*$ and $M_{B_2^*+B_1z}^*$.

We now show that S_1 and S_2 commute.

$$\begin{aligned} S_1^* S_2^* &= W^* M_{B_1^*+B_2z}^* W W^* M_{B_2^*+B_1z}^* W \\ &= W^* M_{B_1^*+B_2z}^* M_{B_2^*+B_1z}^* W \\ &= W^* M_{B_2^*+B_1z}^* M_{B_1^*+B_2z}^* W \\ &= W^* M_{B_2^*+B_1z}^* W W^* M_{B_1^*+B_2z}^* W \\ &= S_2^* S_1^*. \end{aligned}$$

Here we have used the fact that $\text{Ran} W$ is invariant under $M_{B_2^*+B_1z}^*$ and $M_{B_1^*+B_2z}^*$. Therefore, (S_1, S_2, P) is a commuting triple. We now show that (S_1, S_2, P) is a Γ_3 -contraction. If f is a holomorphic polynomial in three-variables, then $f(S_1^*, S_2^*, P^*) = W^* f(T_1^*, T_2^*, V^*) W$, where $(T_1, T_2, V) = (M_{B_1^*+B_2z}, M_{B_2^*+B_1z}, M_z)$. So we have

$$\|f(S_1^*, S_2^*, P^*)\| = \|W^* f(T_1^*, T_2^*, V^*) W\| \leq \|f(T_1^*, T_2^*, V^*)\| \leq \|f\|_{\bar{E}, \infty}.$$

Where the last inequality follows from the fact that (T_1, T_2, V) is a Γ_3 -contraction.

$$\begin{aligned}
S_1^* - S_2 P^* &= W^* M_{B_1^* + B_2 z}^* W - W^* M_{B_2^* + B_1 z} W W^* M_z^* W \\
&= W^* M_{B_1^* + B_2 z}^* W - W^* M_{B_2^* + B_1 z} M_z^* W \\
&= W^* ((I \otimes B_1) + (M_z \otimes B_2^*) - (M_z^* \otimes B_2^*) - (M_z M_z^* \otimes B_1)) W \\
&= W^* (P_C \otimes B_1) W = D_{P^*} B_1 D_{P^*}.
\end{aligned}$$

To obtain the equalities above, we used the fact that $\text{Ran} W$ is invariant under M_z^* . Similarly one can show that $S_2^* - S_1 P^* = D_{P^*} B_2 D_{P^*}$. This shows that (B_1, B_2) is the fundamental operator pair of (S_1^*, S_2^*, P^*) . Let (Y_1, Y_2) be the fundamental operator pair of (S_1, S_2, P) . Then we have, by first part of Theorem 3.9,

$$\begin{aligned}
(B_1^* + B_2 z) \Theta_P(z) &= \Theta_P(z) (Y_1 + Y_2^* z) \text{ and} \\
(B_2^* + B_1 z) \Theta_P(z) &= \Theta_P(z) (Y_2 + Y_1^* z) \text{ holds for all } z \in \mathbb{D}.
\end{aligned}$$

By this and the fact that B_1 and B_2 satisfy equations (3.7) and (3.8), for some operators $A_1, A_2 \in \mathcal{B}(\mathcal{D}_P)$ with numerical radii no greater than 3, we have $A_1 + A_2^* z = Y_1 + Y_2^* z$ and $A_2 + A_1^* z = Y_2 + Y_1^* z$, for all $z \in \mathbb{D}$. Which shows that $Y_1 = A_1$ and $Y_2 = A_2$. Hence (A_1, A_2) is the fundamental operator pair of (S_1, S_2, P) . The proof is now complete. \blacksquare

4. A SET OF UNITARY INVARIANTS

Given two contractions P and P' on Hilbert spaces \mathcal{H} and \mathcal{H}' respectively, we say that the characteristic functions of P and P' coincide if there are unitary operators $u : \mathcal{D}_P \rightarrow \mathcal{D}_{P'}$ and $u_* : \mathcal{D}_{P^*} \rightarrow \mathcal{D}_{P'^*}$ such that the following diagram commutes for all $z \in \mathbb{D}$,

$$\begin{array}{ccc}
\mathcal{D}_P & \xrightarrow{\Theta_P(z)} & \mathcal{D}_{P^*} \\
u \downarrow & & \downarrow u_* \\
\mathcal{D}_{P'} & \xrightarrow{\Theta_{P'}(z)} & \mathcal{D}_{P'^*}
\end{array}$$

In [7], Sz.-Nagy and Foias proved that the characteristic function of a completely non-unitary contraction is a complete unitary invariant. The following theorem says this.

Theorem 4.1. *Two completely non-unitary contractions are unitarily equivalent if and only if their characteristic functions coincide.*

Now we come to the main result of this section which gives a complete set of unitary invariants of pure Γ_3 -contractions.

Theorem 4.2. *Let (S_1, S_2, P) and (S'_1, S'_2, P') be two pure Γ_3 -contractions defined on \mathcal{H} and \mathcal{H}' respectively. Suppose (B_1, B_2) and (B'_1, B'_2) are fundamental operators of (S_1^*, S_2^*, P^*) and (S'^*_1, S'^*_2, P'^*) respectively. Then*

(S_1, S_2, P) is unitarily equivalent to (S'_1, S'_2, P') if and only if the characteristic functions of P and P' coincide and (B_1, B_2) is unitarily equivalent to (B'_1, B'_2) by the same unitary that is involved in the coincidence of the characteristic functions of P and P' .

Proof. Let $U : \mathcal{H} \rightarrow \mathcal{H}'$ be a unitary such that $US_1 = S'_1U, US_2 = S'_2U$ and $UP = P'U$. Then we have

$$UD_P^2 = U(I - P^*P) = U - P'^*PU = D_{P'}^2U,$$

which gives $UD_P = D_{P'}U$. Let $\tilde{U} = U|_{\mathcal{D}_P}$. Then note that $\tilde{U} \in \mathcal{B}(\mathcal{D}_P, \mathcal{D}_{P'})$ and $\tilde{U}D_P = D_{P'}\tilde{U}$. Let (A_1, A_2) and (A'_1, A'_2) be respectively the fundamental operator pairs of (S_1, S_2, P) and (S'_1, S'_2, P') . Then

$$\begin{aligned} D_{P'}\tilde{U}A_1\tilde{U}^*D_{P'} &= \tilde{U}D_PA_1D_P\tilde{U}^* &= \tilde{U}(S_1 - S_2^*P)D_P\tilde{U}^* \\ &= S'_1 - S'^*_2P' = D_{P'}A'_1D_{P'}. \end{aligned}$$

Therefore we have $\tilde{U}A_1\tilde{U}^* = A'_1$. Similarly one can prove that $\tilde{U}A_2\tilde{U}^* = A'_2$.

Let $u : \mathcal{D}_P \rightarrow \mathcal{D}_{P'}$ and $u_* : \mathcal{D}_{P^*} \rightarrow \mathcal{D}_{P'^*}$ be unitary operators such that

$$u_*B_1 = B'_1u_*, u_*B_2 = B'_2u_* \text{ and } u_*\Theta_P(z) = \Theta_{P'}(z)u \text{ holds for all } z \in \mathbb{D}.$$

The unitary operator $u_* : \mathcal{D}_{P^*} \rightarrow \mathcal{D}_{P'^*}$ induces a unitary operator $U_* : H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*} \rightarrow H^2(\mathbb{D}) \otimes \mathcal{D}_{P'^*}$ defined by $U_*(z^n \otimes \xi) = (z^n \otimes u_*\xi)$ for all $\xi \in \mathcal{D}_{P^*}$ and $n \geq 0$. Note that

$$U_*(M_{\Theta_P}f(z)) = u_*\Theta_P(z)f(z) = \Theta_{P'}(z)uf(z) = M_{\Theta_{P'}}(uf(z)),$$

for all $f \in H^2(\mathbb{D}) \otimes \mathcal{D}_P$ and $z \in \mathbb{D}$. Hence U_* takes $\text{Ran}M_{\Theta_P}$ onto $\text{Ran}M_{\Theta_{P'}}$. Since U_* is unitary, we have

$$U_*(\mathcal{H}_P) = U_*((\text{Ran}M_{\Theta_P})^\perp) = (U_*\text{Ran}M_{\Theta_P})^\perp = (\text{Ran}M_{\Theta_{P'}})^\perp = \mathcal{H}_{P'}.$$

By definition of U_* we have

$$\begin{aligned} U_*(I \otimes B_1^* + M_z \otimes B_2)^* &= (I \otimes u_*)(I \otimes B_1 + M_z^* \otimes B_2^*) \\ &= I \otimes u_*B_1 + M_z^* \otimes u_*B_2^* \\ &= I \otimes B'_1u_* + M_z^* \otimes B'^*_2u_* \\ &= (I \otimes B'_1 + M_z^* \otimes B'^*_2)(I \otimes u_*) \\ &= (I \otimes B'^*_1 + M_z \otimes B'_2)^*U_*. \end{aligned}$$

Similar calculation gives us

$$U_*(I \otimes B_2^* + M_z \otimes B_1)^* = (I \otimes B'^*_2 + M_z \otimes B'_1)^*U_*.$$

Therefore $\mathcal{H}_{P'} = U_*(\mathcal{H}_P)$ is a co-invariant subspace of $(I \otimes B'^*_1 + M_z \otimes B'_2)$ and $(I \otimes B'^*_2 + M_z \otimes B'_1)$. Hence $P_{\mathcal{H}_P}(I \otimes B_1^* + M_z \otimes B_2)|_{\mathcal{H}_P}$ and $P_{\mathcal{H}_{P'}}(I \otimes B'^*_1 + M_z \otimes B'_2)|_{\mathcal{H}_{P'}}$ are unitarily equivalent and so are also $P_{\mathcal{H}_P}(I \otimes B_2^* + M_z \otimes B_1)|_{\mathcal{H}_P}$ and $P_{\mathcal{H}_{P'}}(I \otimes B'^*_2 + M_z \otimes B'_1)|_{\mathcal{H}_{P'}}$. It is evident that the unitary operator

that intertwines them is $U_*|_{\mathcal{H}_P} : \mathcal{H}_P \rightarrow \mathcal{H}_{P'}$.

Also we have

$$U_*(M_z \otimes I_{\mathcal{D}_{P^*}}) = (I \otimes u_*)(M_z \otimes I_{\mathcal{D}_{P^*}}) = (M_z \otimes I_{\mathcal{D}_{P'^*}})(I \otimes u_*) = (M_z \otimes I_{\mathcal{D}_{P'^*}})U_*.$$

So $P_{\mathcal{H}_P}(M_z \otimes I_{\mathcal{D}_{P^*}})|_{\mathcal{H}_P}$ and $P_{\mathcal{H}_{P'}}(M_z \otimes I_{\mathcal{D}_{P'^*}})|_{\mathcal{H}_{P'}}$ are unitarily equivalent by the same unitary $U_*|_{\mathcal{H}_P}$. Therefore (S_1, S_2, P) and (S'_1, S'_2, P') are unitarily equivalent. Hence we are done. ■

REFERENCES

- [1] J. Agler and N. J. Young, A commutant lifting theorem for a domain in \mathbb{C}^2 and spectral interpolation, *J. Funct. Anal.* 161 (1999), 452 – 477.
- [2] J. Agler and N. J. Young, A model theory for Γ -contractions, *J. Operator Theory* 49 (2003), 45-60.
- [3] T. Bhattacharyya, S. Pal and S. Shyam Roy, Dilations of Γ -contractions by solving operator equations, *Adv. Math.* 230 (2012), 577 – 606.
- [4] T. Bhattacharyya and S. Pal, A functional model for pure Γ -contractions, *J. Operator Theory*, 71 (2014), 327 – 339.
- [5] S. Biswas and S. Shyam Roy, Functional models for Γ_n -contractions and characterization of Γ_n -isometries, *J. Func. Anal.*, 266 (2014), 6224 – 6255.
- [6] T. Gamelin, Uniform Algebras, *Prentice-Hall, New Jersey*, 1969.
- [7] B. Sz.-Nagy, C. Foias, H. Bercovici and L. Kerchy, Harmonic analysis of operators on Hilbert space, Universitext, *Springer, New York*, 2010.
- [8] S. Pal, From Stinespring dilation to Sz.-Nagy dilation on the symmetrized bidisc and operator models, *New York Jour. Math.*, 20 (2014), 545 – 564.
- [9] S. Pal, Rational dilation on the symmetrized tridisc: failure, success and unknown, *Preprint*.
- [10] S. Pal and O. M. Shalit, Spectral sets and distinguished varieties in the symmetrized bidisc, *J. Funct. Anal.*, 266 (2014), 5779 – 5800.
- [11] S. Pal and S. Roy, A Schwarz lemma for the symmetrized tridisc and description of interpolating functions, *Preprint*.
- [12] F. H. Vasilescu, Analytic Functional Calculus and Spectral Decompositions, *Editura Academiei: Bucuresti, Romania and D. Reidel Publishing Company*, 1982.

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